Journal of Statistical Physics, Vol. 78, Nos. 3/4, 1995

Classification of the Similarity Solutions of Free Kramers Equation

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Received July 20, 1994

We obtain a complete classification of all possible nontrivial similarity solutions of the free Kramers equation, together with a necessary and sufficient condition for each type to be reducible to the heat equation. A confluent hypergeometric solution of the free Kramers equation is derived for some classes of similarity solutions.

KEY WORDS: Brownian motion; free Kramers type; group classification; heat and confluent hypergeometric solution.

1. INTRODUCTION

The unidimensional Brownian motion of a gas can be described reasonably well by the Fokker–Planck equation of free Kramers type

$$\frac{\partial u}{\partial t}(x, y, t) + y\frac{\partial u}{\partial x}(x, y, t) - r\frac{\partial}{\partial y}\left(y + \frac{K_B T}{M}\frac{\partial}{\partial y}\right)u(x, y, t) = 0 \qquad (1)$$

$x \in R$, $y \in R$. Here K_B is the Boltzmann constant and r is the friction coefficient, depending on the size and mass M of the particles; T is temperature.[1]

As the distribution function, u must be a nonnegative function of its physical coordinates x, y, t and parameters $K_B T/M = R$ and r, i.e., $u(x, y, t; r, R) \geqslant 0$; for simplicity we choose the units so that $r = R = 1$. The arbitrariness of the parameters r and R does not alter the analytic equations; we return to this point in the concluding discussion.

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0022-4715/95/0200-1139$07.50/0

Table I. Similarity Forms and Reduced Kramers Equations

Case	Similarity variables	Reduced Kramers equation
B_4	$s=t, z=x-yt,$ $u=\exp[-y^2(s+1)/4-yz/2]\,F$	$F_s=s^2F_{zz}+szF_z+[(s+1)/2+z^2/4]\,F$
B_5	$s=t, z=x-y, u=\exp(-y^2/2)\,F$	$F_s=F_{zz}$
B_6	$s=t, z=x+y, u=F$	$F_s=F_{zz}+F$
B_1+B_2	$s=t-x, z=y, u=F$	$F_s=(F_{zz}/1-z)+(zF_z/1-z)+(F/1-z)$
B_1+B_4	$s=(t^2-2x)/2, z=t-y$ $u=e^hF, h=(t^2/4)+(t^3/6)-(xt+yt)/2$	$zF_s=F_{zz}+(z-1)F_z+\frac{1}{2}(2-s-z)\,zF$
B_2+B_5	$s=t, z=-y+(x-y)e^t, u=e^hF,$ $h=-y^2/2$	$F_s=(e^s+1)^2F_{zz}+zF_z$
B_2+B_6	$s=t, z=y+(x+y)\,e^{-t}, u=F$	$F_s=(e^{-s}+1)^2F_{zz}+zF_z+F$
B_4+B_5	$s=t, z=x-yt+(x-y)e^t, u=e^hF$ $h=[2yz+y^2(1+t+2e^{2t}+4e^t)]/[-4(e^t+1)^2]$	$F_s=(s+e^s)^2F_{zz}+\dfrac{s-e^{2s}}{(1+e^s)^2}zF_z+\dfrac{z^2+2(1+e^s)(1+s+2e^s)}{4(1+e^s)^4}F$
B_5+B_6	$s=t, z=x-y-e^{-2t}(x+y), u=e^hF,$ $h=(-y^2)/2(1-e^{-2t})$	$F_s=(1+e^{-2s})^2F_{zz}-\dfrac{2e^{-2s}}{1-e^{-2s}}zF_z-\dfrac{e^{-2s}}{1-e^{-2s}}F$
B_4+B_6	$s=t, z=x-yt-(x+y)\,e^{-t}, u=e^hF,$	$F_s=(s+e^{-s})^2F_{zz}+\dfrac{s+e^{-2s}}{(1-e^{-s})^2}zF_z$
	$h=(2yz+y^2+ty^2)/[-4(1-e^{-t})^2$	$+\left(1+\dfrac{z^2}{4(1-e^{-s})^4}+\dfrac{s+2e^{-s}-1}{2(1-e^{-s})^2}\right)F$
$B_4+B_5+B_6$	$s=t, z=x-yt+e^t(x-y)-e^{-t}(x+y), u=e^hF$	$F_s=(s+e^s+e^{-s})^2F_{zz}+\left(\dfrac{s-2e^{-s}-e^{2s}+e^{-2s}}{(1+e^s-e^{-s})^2}\right)zF_z$
	$h=[2yz+y^2(t-1+4e^t+2e^{2t}]/[-4(1+e^t-e^{-t})^2$	$+\left(\dfrac{z^2}{4(1+e^s-e^{-s})^4}+\dfrac{s-1+2e^s-2e^{-s}-2e^{-2s}}{2(1+e^s+e^{-s})^2}\right)F$

The most extended Lie group of tranformations admitted by Eq. (1) depends on six arbitrary group constants.[2] It gives rise to the trivial symmetries generated by

$$B_1 = \frac{\partial}{\partial t}, \qquad B_2 = \frac{\partial}{\partial x}, \qquad B_3 = u\frac{\partial}{\partial u}, \qquad B_g = g(x, y, t)\frac{\partial}{\partial u}$$

Here g is an arbitrary solution of Eq. (1). The symmetry B_g is characteristic for the linear nature of Eq. (1) and demonstrates the superposition principle of linear equations. The only three nontrivial symmetry generators are given by

$$\begin{aligned} B_4 &= t\partial_x + \partial_y - \tfrac{1}{2}(x+y)\,u\partial_u \\ B_5 &= (\partial_x + \partial_y - yu\partial_u)e^t \\ B_6 &= (\partial_x - \partial_y)\,e^{-t} \end{aligned} \tag{2}$$

In the present work, we propose a natural classification of the possible similarity solutions admitted by the free Kramers equation (1).

Precisely, by introducing the adjoint representation of the Lie algebra,[3] we obtain the basic fields of an optimal system, from which every other solution can be derived, given 11 combinations of symmetries. This produces the essential types of the reduced Kramers equation, which are partial differential equations of the similarity variables S and Z, as well as similarity solutions $F(S, Z)$ found by a systematic use of the Lie similarity method.[3-5] One finds the 11 essential subgroups listed in Table I. One obtains further solutions of Eq. (1) by applying finite group transformations to these essential solutions.[3]

In the following, we look for transformations which will map the reduced Kramers equations in Table I to some type that is known to have a fundamental solution, from which the analytical solution of the free Kramers equation may be constructed. We obtain the two following cases.

2. CONNECTION TO THE HEAT EQUATION

To see how this relation appears in the reduced Kramers equations in Table 1, except those that correspond to the subclasses $B_1 + B_2$ and $B_1 + B_4$, one can write them in the form

$$F_S = a(S, Z)\,F_{ZZ} + b(S, Z)\,F_Z + d(S, Z)F \tag{3}$$

Here and in the following subscripts means differentiation.

If we let $S = V$ and $W = R(S, Z)$, where

$$R(S, Z) = \int_0^Z (a(s, n))^{-1/2}\, dn \tag{4}$$

then Eq. (3) becomes

$$F_V = F_{WW} + D(V, W) F_W + d(V, W) F \tag{5}$$

where

$$D(V, W) = -R_S + a(S, Z)\, R_{ZZ} + b(S, Z) R_Z$$

Now let $C(V, W)$ and $G(V, W)$ be such that

$$G = e^{-C} F \tag{6}$$

and

$$C_W = -\tfrac{1}{2} D(V, W) \tag{7}$$

We find that Eq. (5) is equivalent to

$$G_V = G_{WW} + K(V, W) G \tag{8}$$

where

$$K(V, W) = -C_V + C_{WW} - C_W^2 + d(V, W) \tag{9}$$

Bluman[6] proved that if $K(V, W)$ is of the form

$$K(V, W) = q_0(V) + q_1(V)\, W + q_2(V)\, W^2 \tag{10}$$

where q_0, q_1, and q_2 are arbitrary functions of V, we can transform Eq. (8) to the heat equation constructively, by a one-to-one mapping.

Fortunately, making use of transformations (4), (6) and (7) the reduced Kramers equations in Table I (except those that corresponds to $B_1 + B_2$ and $B_1 + B_4$) will transform to equations of the form (8) with $K(V, W)$ satisfying the condition in (10).

3. CONFLUENT HYPERGEOMETRIC SOLUTION

Looking for exact solutions of the PDE corresponding to the subclasses $B_1 + B_2$ and $B_1 + B_4$, we tried the Lie group method, which leads to the variable separable form of solution. For the first one, substitute $F(S, Z) = e^{S - Z} L(Z)$. The corresponding PDE transforms to the ODE

$$d^2L/dz^2 + (z - 2)\, dz + L = 0 \tag{11}$$

Using the transformation $h = -(z-2)^2/2$, one obtains

$$h\, d^2L/dh^2 + (\tfrac{1}{2} - h)\, dL/dh - \tfrac{1}{2}L = 0 \tag{12}$$

which is a confluent hypergeometric equation. It has the solution

$$L(h) = {}_1F_1(\tfrac{1}{2}, \tfrac{1}{2}, h) \tag{13}$$

Similarly, for the second subclass $B_1 + B_4$, let $F(s, z) = e^{s - s^2/4} N(z)$; then the corresponding PDE transform to

$$d^2N/dz^2 + (z-1)\, dN/dz - \tfrac{1}{2}z^2 N = 0 \tag{14}$$

which may be written as

$$d^2L/dz^2 + (\sqrt{3}z - 1/\sqrt{3})\, dL/dz + (\sqrt{3}/2 - 3/2)L = 0 \tag{15}$$

where

$$L(z) = \exp(Kz^2 - 3kz/\sqrt{3})\, N(z)$$

and K satisfies $4K^2 + 2k - 1/2 = 0$. Equation (15) can be written as a confluent hypergeometric equation

$$h\, d^2L/dh^2 + (1/2 - h)\, dL/dh - (9/2 - 2\sqrt{3})L = 0 \tag{16}$$

where

$$h = -(\sqrt{3}z - 1/\sqrt{3})^2/2\sqrt{3}$$

4. CONCLUDING DISCUSSION

We conclude by discussing two specific comments:

1. It is important to emphasize that the occurrence of the physical parameters R and r in the symmetry generators does not alter the analytic form of the reduced Kramers equations in Table I.

In general, the explicit dependence of the symmetry generators on parameters R and r corresponding to Eq. (2) is

$$\begin{aligned} B_4 &= t\partial_x + \partial_y - (1/2R)(y + rx)\, u\partial_u \\ B_5 &= e^{rt}[(1/r)\partial_x + \partial_y - (1/R)\, yu\partial_u] \\ B_6 &= e^{rt}[(1/r)\partial_x - \partial_y] \end{aligned} \tag{17}$$

The vectors B_1, B_2, and B_3 will not change.

2. For the free Kramers equation (1), where we seek a similarity solution of the distribution function u for which

$$u(x, y, t; r, R) \geqslant 0 \tag{18}$$

two remarks are in order. First, it should be emphasized that the distribution function u is connected with the similarity solution $F(s, z)$ of the reduced equation by relation of the form $u = e^h F$. Second, these reduced Kramers equations transform to the heat equation or the confluent hypergeometric equation, where the nonnegative solutions are satisfied.

To incorporate these comments, let us consider the following illustrative examples.

(i) The group-invariant B_5 suggests the similarity variables s and z and similarity solution $F(s, z)$ to be $s = t$, $z = rx - y$, and $u = \exp(-y^2/2R) F$. Substitution of the similarity into Eq. (1) results in $F_s = rRF_{zz}$, which has the fundamental solution

$$F(s, z) = (4rR\pi s)^{-1/2} \exp(-z^2/4rRs) \tag{19}$$

If we invert all our previously used transformations, the distribution function is

$$u(x, y, t; r, R) = (4rR\pi t)^{-1/2} \exp[-y^2/2R - (rx - y)^2/4rRt] \tag{20}$$

(ii) The similarity representations corresponding to the vector field B_6 are $s = t$, $z = rx + y$, and $u = F(s, z)$. The reduced equation (1) reads $F_s = rRF_{zz} + rF$, which can be transformed to the heat equation $G_s = rRG_{zz}$ by the transformation $F = e^{rs} G$. Then, Eq. (1) has the solution

$$u(x, y, t; r, R) = (4rR\pi t)^{-1/2} \exp[rt - (rx + y)^2/4rRt] \tag{21}$$

(iii) For the class of similarity solutions generated by the vector field B_4, the general reduction can be obtained by the similarity representation $s = t$, $z = x - yt$, and $u = \exp(-h) F$, where $h = [(1 + rs) y^2 + 2rzy]/4R$. This solution inserted in Eq. (1) gives

$$F_s = rRs^2 F_{zz} + r^2 sz F_z + [2rR(rs + 1) + r^3 z^2]/4RF \tag{22}$$

Making use of transformations (4), (6), and (7), we get

$$G_v = G_{ww} + K(v, w) G \tag{23}$$

where

$$s = v, \qquad w = (rR)^{-1/2} z/s, \qquad G(v, w) = \exp[w^2(1 + r^2 v^2)/4v]\, F(v, w)$$

and

$$K(v, w) = q_0(v) + q_2(v) w^2 = (rv - 1)/2v - [(r^2v^2 + 2)/4v^2] w^2 \quad (24)$$

Equations of the form (23) can be transformed to the heat equation

$$\bar{G}_{\bar{v}} = \bar{G}_{\bar{w}\bar{w}} \quad (25)$$

by a one-to-one mapping (for details see ref. 4, Chapter 6, where a specific algorithm to determine this mapping is established),

$$\bar{v} = V(v), \qquad \bar{w} = W(v, w) = w(V')^{1/2}, \qquad \bar{G} = \exp(w^2 V''/8V')\, R(v) \quad (26)$$

where

$$R(v) = \exp\left[\int_0^v - m(s)/n(s)\, ds\right], \qquad m = -n'/4 - q_0 n, \qquad n = 2V/V'$$

and V is determined by the relation

$$2M' = M^2 + 16q_2 \quad (27)$$

where

$$M = V''/V'$$

In a similar fashion, the other reduced Kramers equations can be transformed to the heat equation and this ensures that the solutions are nonnegative.

(iv) As an example of the other class of the reduced Kramers equations, which transform to the confluent hypergeometric equation, consider the subclass $B_1 + B_2$, where the solution of Eq. (12) is

$$L(h) = {}_1F_1(1/2, 1/2, h) = e^h \qquad \text{and} \qquad 2h = -(z - 2)^2$$

One obtains

$$u(x, y, t) = \exp(t - x - 2 + y - y^2/2) \quad (28)$$

As a final comment we conclude that the great variety of solutions to Eq. (1) obtained in Table I allows a group classification and presents a new class of solutions which are not equivalent to the heat equation.

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Communicated by J. L. Lebowitz